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Numerical Simulation of Transport in a Field-Reversed Mirror Plasma

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A computer code is described which simulates the evolution of the axisymmetric toroidal region of a field-reversed mirror plasma. The code alternates between a 1–D transport calculation and a 1–D or 2–D equilibrium calculation. Four 1–D transport equations are solved simultaneously for the ion density, electron entropy, ion entropy and toroidal magnetic flux. The transport equations for these adiabatic quantities are independent of the time rate of change of the poloidal magnetic flux. This choice of dependent variables eliminates some of the coupling of the transport and equilibrium calculation. The physical processes which are modeled by the transport calculation are classical transport using Braginskii transport coefficients, Joule heating of the electrons, collisional transfer of energy between ion and electrons, charge exchange loss of ion energy, radiation cooling of electrons due to impurities, heating of ions by neutral beams and enhancement of electron thermal conductivity by a given factor. The equilibrium calculation consists of the solution of the 2–D Grad–Shafranov equation in the r, z grid. Or, if the adiabatic quantities have not changed much, the 1–D flux surface averaged Grad–Shafranov equation is solved.

1. INTRODUCTION

This paper describes a computer code for the calculation of the evolution of a field-reversed axisymmetric plasma. The plasma is assumed to occupy the region inside the separatrix. The magnetic field lines inside the separatrix are closed and form nested toroidal srfaces. Outside the separatrix (Fig. 1) the field lines are open. The magnetic field is reversed in the sense that the magnetic field on the axis of rotational symmetry, $r = 0$, inside the plasma is in the direction opposite to the external guide magnetic field.

This code differs from most tokamak transport codes in that the independent variables is proportional to the poloidal magnetic flux instead of the toroidal magnetic flux. The poloidal flux is used because some of the simulations have zero toroidal magnetic fields. The computational region of this code includes a separatrix which includes $r = 0$.

The code is used to model the Beta II magnetized coaxial plasma gun experiment at Lawrence Livermore National Laboratory. In the Beta II experiment a toroidal plasma is injected into a flux conserving cylinder from a plasma gun. The code describes the subsequent evolution of the plasma. The flux conserver used in this

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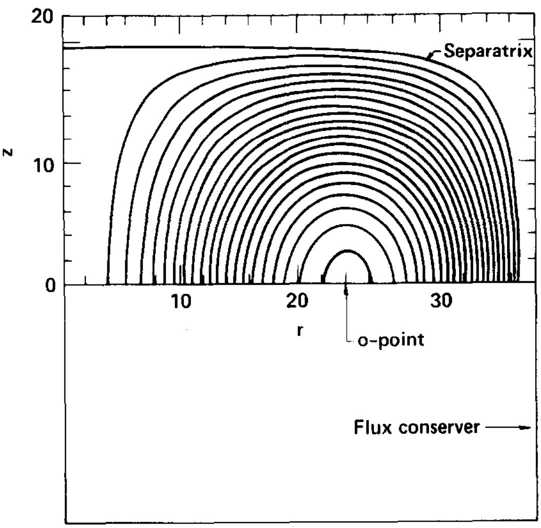

FIG. 1. Flux surface structure in the r, z plane. Symmetry is assumed about the $z = 0$ plane.

simulation is a closed can which is symmetric about $z = 0$. In the actual experiment the can has a hole in one end through which the plasma enters. The axisymmetric magnetic field can be written as

$$\mathbf{B} = \nabla\psi \times \nabla\theta + f\,\nabla\theta. \tag{1}$$

The toroidal angle is θ. Inside the separatrix the surfaces of constant ψ form closed troidal surfaces. f is the toroidal magnetic field times the radius r.

With an axisymmetric magnetic field and with the assumption that the plasma is near thermodynamic equilibrium, it can be shown that the particle density, N, and the temperatures, T_e and T_i, are functions of ψ and time t only [1]. Also the toroidal magnetic field function, f, is a function of ψ and t only [1]. The plasma is assumed to be in a state of approximate force balance. The lowest order momentum equation for the plasma is

$$\nabla P = \frac{\mathbf{J} \times \mathbf{B}}{c}. \tag{2}$$

The ψ function which describes the equilibrium is the solution of the Grad–Shafranov equation [2].

$$\nabla \cdot \left(\frac{\nabla\psi}{r^2}\right) = -4\pi \frac{dP}{d\psi} - \frac{f}{r^2}\frac{df}{d\psi} \tag{3}$$

which is obtained from Eqs. (1) and (2). The pressure, P, in this equation is the sum of the ion and electron pressure. Since the plasma is assumed to be near a thermodynamic equilibrium, the evolution of the plasma parameters can be described by a diffusion equation. There are four plasma parameters which are advanced in time with the FRT code (Field Reversed Transport). These quantities are

$$Q_1 = N_i S_1, \qquad Q_2 = P_e S_1^{5/3}, \tag{4a}$$

$$Q_3 = P_i S_1^{5/3}, \qquad Q_4 = f S_2. \tag{4b}$$

The plasma is assumed to consist of only ion and electrons. The S_1 and S_2 in the above equations are integrals over the closed flux surfaces.

$$S_1 = \int \frac{d^2 r}{|\nabla \rho|}, \tag{5}$$

$$S_2 = \int \frac{d^2 r}{|\nabla \rho|} \frac{1}{r^2}. \tag{6}$$

The independent variable for the transport equation is

$$\rho = \frac{\psi_m - \psi}{\psi_\delta}, \tag{7a}$$

where

$$\psi_\delta = \psi_m - \psi_b. \tag{7b}$$

Where ψ_m is the value of ψ on the magnetic axis, and ψ_b is the ψ value on the boundary of the plasma, the separatrix.

The quantities Q_1, Q_2, Q_3 and Q_4 were chosen as dependent variables because they are adiabatic quantities. That is, if there is no diffusion, these quantities are constants. The first of these quantities, Q_1, is proportional to the number of ions between adjacent flux surfaces. A transport equation for the electrons is not needed since, due to quasineutrality, the electron density is equal to the ion density. The Q_2 and Q_3 are proportional to the electron and ion entropy between adjacent flux surfaces. The fourth quantity, Q_4, is proportional to the toroidal magnetic flux enclosed between adjacent flux surfaces.

The advantage of using these quantities for dependent variables is that the transport equations do not depend on the flux surface velocity, which is proportional to $\partial\psi/\partial t$. This simplifies the iteration in the solution of the transport equations.

Boundary conditions are applied to transport equations at the separatrix ($\rho = 1$). The boundary conditions are based on the assumption that the particle density and temperatures are given at the separatrix. Also the toroidal magnetic flux function, f, is set to zero on the separatrix. The boundary conditions at the o-point are that the particle and heat fluxes are zero.

The physical process is envisioned as proceeding in two alternating steps:

(1) *Non-adiabatic step.* The diffusion process occurs. A set of 1–D transport equations advance the quantities Q_1, Q_2, Q_3 and Q_4.

(2) *Adiabatic step.* The quantities Q_1, Q_2, Q_3 and Q_4 are assumed to be constant. The 2–D equilibrium equation (3) is solved, or the flux surface averaged Grad–Shafranov equation is solved.

The plasma properties are advanced by this "alternating dimension method." A review of this method is given by Grad [2]. Similar $1\frac{1}{2}$ D transport codes have been used to study transport in tokomaks [3–6].

In Section II the transport equations are derived. They are obtained by averaging the transport equations of Braginskii [7] over a surface of constant ψ. The method of solving the set of transport equations are given in that section. In Section III the method of solving the Grad–Shafranov equation is given [8]. In Section IV the coupling of the equilibrium and transport calculation is presented. Some results of the FRT code are given in Section V.

II. Transport Calculations

A. *Introduction*

In this section the 1–D transport equations will be derived by averaging the 3–D Braginskii transport equations over a surface of constant ρ. The plasma is assumed to consist of hydrogen (or deuterium).

In the experiment processes other than classical transport are important. The computer code attempts to account for this by including models for other processes. Radiation cooling of electrons due to the presence of impurities is included by using the data by Post *et al.* [9]. The impurity density is assumed to be a given function of ρ. Anomalous transport of electron energy is accounted for by increasing the electron thermal conductivity by some given factor. Neutral beam injection is included as a source in the ion entropy equation [10].

In this section the 1–D transport equations for the four quantities Q_1, Q_2, Q_3 and Q_4 are derived. These four equations are written in the form

$$\frac{dQ_l}{dt} = \mathscr{S}_l \left[-\frac{d}{d\rho} \left\{ \sum_{k=1}^{4} B_{l,k} \frac{d}{d\rho} (A_k Q_k) + C_l + D_l A_l Q_l \right\} + \sum_{k=1}^{4} \left\{ E_{l,k} \frac{d}{d\rho} (A_k Q_k) + F_{l,k} A_k Q_k \right\} + G_l \right], \qquad l = 1, 2, 3, 4, \tag{8}$$

where the transport coefficients $B_{l,k}$, C_l, D_l, $E_{l,k}$, $F_{l,k}$ and G_l will be derived in this section.

The constants A_k are

$$A_1 = S_1^{-1}, \qquad A_2 = S_1^{-5/3}, \tag{9a}$$

$$A_3 = S_1^{-5/3}, \qquad A_4 = S_2^{-1}, \tag{9b}$$

so that

$$N_i = Q_1 A_1, \qquad P_e = Q_2 A_2, \tag{9c}$$

$$P_i = Q_3 A_3, \qquad f = Q_4 A_4. \tag{9d}$$

The $\mathscr{S}$'s are

$$\mathscr{S}_1 = 1, \qquad \mathscr{S}_2 = S_1^{2/3}, \tag{9e}$$

$$\mathscr{S}_3 = S_1^{2/3}, \qquad \mathscr{S}_4 = 1. \tag{9f}$$

B. *Particle Transport Equation*

The 1–D particle transport equation is derived from the particle conservation equation.

$$\frac{\partial N_i}{\partial t} + \nabla \cdot (N_i u_i) = 0, \tag{10}$$

where the ion fluid velocity is u_i. The component of u_i which is needed in the averaged particle transport equation is obtained from the first order ion momentum equation. The zeroth order momentum equations are

$$\nabla P_e = -eN_e \left[\frac{u_e \times \mathbf{B}}{c}\right], \tag{11a}$$

$$\nabla P_i = eN_i \left[\frac{u_i \times \mathbf{B}}{c}\right]. \tag{11b}$$

These equations contain the large component of u_e and u_i which are in the plane of the flux surface and do not contribute directly to the transport of particles across a flux surface. The sum of these two equations, (11a), (11b), will yield the pressure balance equation (2).

The next order momentum equation will yield the u_i need in the transport equation (8). This equation is

$$eN_i \left[\mathbf{E} + \frac{u_i \times \mathbf{B}}{c}\right] = \mathbf{R}. \tag{12}$$

Any contribution due to an anisotropic pressure tensor, $\nabla \cdot \mathbf{P}_i$, has been ommitted. If included, this term would lead to a neo-classical transport term which would require a kinetic theory treatment.

The momentum transfer term given by Braginskii consist of two terms.

$$\mathbf{R} = \mathbf{R}_u + \mathbf{R}_T. \tag{13}$$

The first term, $\mathbf{R}_u$, is due to the relative velocity between ions and electrons.

$$\mathbf{R}_u = eN_i \left[\frac{J_{\parallel}}{\sigma_{\parallel}} + \frac{J_{\perp}}{\sigma_{\perp}} \right] \tag{14}$$

The other term is the thermal force contribution.

$$\mathbf{R}_T = -\frac{3}{2} \frac{N_i}{\omega_e \tau_e} \mathbf{b} \times \nabla T_e. \tag{15}$$

The unit vector in the direction of the magnetic field is **b**. The electrical conductivities are

$$\sigma_{\parallel} = 1.96\sigma_{\perp}, \qquad \sigma_{\perp} = \frac{e^2 N_i \tau_e}{m_e}. \tag{16}$$

The electron and ion gyrofrequencies are

$$\omega_e = \frac{eB}{m_e c}, \qquad \omega_i = \frac{eB}{m_i c}. \tag{17}$$

The electron and ion collision frequencies are

$$\tau_e = \frac{3 m_e^{1/2} T_e^{3/2}}{4(2\pi)^{1/2} \ln \Lambda e^4 N_i}, \tag{18a}$$

$$\tau_i = \frac{3 m_i^{1/2} T_i^{3/2}}{4\pi^{1/2} \ln \Lambda e^4 N_i}, \tag{18b}$$

Since the magnetic field structure is assumed to be given during the transport calculation, the parallel and perpendicular currents are also given,

$$\mathbf{J}_{\parallel} = \left[\frac{cB}{4\pi} \frac{df}{d\psi} + \frac{cf}{B} \frac{dP}{d\psi} \right] \mathbf{b}, \tag{19}$$

$$\mathbf{J}_{\perp} = \frac{c}{B} \mathbf{b} \times \nabla P. \tag{20}$$

The time derivative at constant ρ is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}_\rho \cdot \nabla, \tag{21}$$

where $\mathbf{V}_\rho$ is the velocity of a surface of constant ρ. It is related to $\dot{\rho}$, $(\cdot = \partial/\partial t)$, by

$$\mathbf{V}_\rho \cdot \nabla\rho = -\dot{\rho} \tag{22a}$$

or in terms of $\dot{\psi}$,

$$\mathbf{V}_\rho \cdot \nabla\rho = \frac{\dot{\psi} - \dot{\psi}_m(1-\rho) - \dot{\psi}_b\rho}{\psi_\delta}. \tag{22b}$$

The particle conservation equation (10) is integrated over a surface of constant ρ, using Eq. (21)

$$\int \frac{d^2r}{|\nabla\rho|}\frac{dN_i}{dt} - \int \frac{d^2r}{|\nabla\rho|}\mathbf{V}_\rho \cdot \nabla N_i = -\frac{d}{d\rho}\int d^3r\,\nabla\cdot(N_i u_i). \tag{23}$$

The second term on the left-hand side is changed into a volume integral then integrated by parts. The right-hand side is also integrated by parts. The surface integral term from the integration by parts of the second term on the left-hand side is combined with the term on the right-hand side. Using the relation,

$$\int \frac{d^2r}{|\nabla\rho|}\nabla\cdot\mathbf{V}_\rho = \frac{d}{dt}\int\frac{d^2r}{|\nabla\rho|}, \tag{24}$$

Eq. (23) becomes

$$\frac{d}{dt}\left\{N_i\int\frac{d^2r}{|\nabla\rho|}\right\} = -\frac{d}{d\rho}\int\frac{d^2r}{|\nabla\rho|}N_i\,\nabla\rho\cdot(u_i - \mathbf{V}_\rho). \tag{25}$$

The right-hand side of this equation is obtained by taking the dot product of the momentum euation (12) with $\nabla\psi\times\mathbf{B}$. By using the form of the magnetic field, (1), and the θ component of Faradays's Law the $\mathbf{E}\times\mathbf{B}\cdot\nabla\rho$ term can be written as

$$\mathbf{E}\times\mathbf{B}\cdot\nabla\psi = -f\mathbf{E}\cdot\mathbf{B} - \frac{B^2}{c}\frac{\partial\psi}{\partial t}. \tag{26}$$

Using Eq. (26) the momentum equation becomes

$$(u_i - \mathbf{V}_\rho)\cdot\nabla\rho = \frac{c}{B^2\psi_\delta}\left\{f\mathbf{E}\cdot\mathbf{B} + \frac{B^2}{c}\left[\rho\dot{\psi}_b + (1-\rho)\,\dot{\psi}_m\right] + \frac{\mathbf{R}\times\mathbf{B}\cdot\nabla\psi}{eN_i}\right\}. \tag{27}$$

The $\mathbf{E}\cdot\mathbf{B}$ term is obtained by taking the dot product of the momentum equation (12) with $\mathbf{B}$. The $\mathbf{R}\times\mathbf{B}\cdot\nabla\psi$ term is evaluated using Eq. (13). The particle transport equation is then

$$\begin{aligned}\frac{d}{dt}(N_iS_1) = -\frac{d}{d\rho}\bigg\{\frac{N_ic^2}{\psi_\delta^2}\bigg[&-\frac{fS_1}{4\pi\sigma_{\|}}\frac{df}{d\rho} - \frac{f^2S_4}{\sigma_{\|}}\left(\frac{dP_e}{d\rho} + \frac{dP_i}{d\rho}\right)\\ &-\frac{S_3}{\sigma_\perp}\left(\frac{3}{2}T_e\frac{dN_i}{d\rho} - \frac{1}{2}\frac{dP_e}{d\rho} + \frac{dP_i}{d\rho}\right)\\ &+\frac{\psi_\delta S_1}{c^2}\left(\dot{\psi}_m(1-\rho) + \dot{\psi}_b\rho\right)\bigg]\bigg\}.\end{aligned} \tag{28}$$

The additional surface integral which appear in this equation are

$$S_3 = \int \frac{d^2r}{|\nabla\rho|} \frac{|\nabla\psi|^2}{B^2}, \tag{29}$$

$$S_4 = \int \frac{d^2r}{|\nabla\rho|} \frac{1}{B^2}. \tag{30}$$

C. *Electron Entropy Ttransport Equation*

An equation will be derived in this section to advance $Q_2 S_1^{5/3}$ which is proportional to the electron entropy. This equation is derived from the electron temperature equation of Braginskii,

$$\frac{3}{2} N_i \left[\frac{\partial T_e}{\partial t} + u_e \cdot \nabla T_e\right] + P_e \nabla \cdot u_e = -\nabla \cdot \mathbf{q}_e + Q_e, \tag{31}$$

where $\mathbf{q}_e$ and Q_e are given by

$$\mathbf{q}_e = \frac{3}{2} \frac{N_i T_e}{\omega_e \tau_e} \mathbf{b} \times u_e - 4.66 \frac{N_i T_e}{m_e \omega_e^2 \tau_e} \nabla T_e - \frac{5}{2} \frac{c N_i T_e}{eB} \mathbf{b} \times \nabla T_e, \tag{32}$$

$$Q_e = \frac{J_{\|}^2}{\sigma_{\|}} + \frac{J_{\perp}^2}{\sigma_{\perp}} + \frac{\mathbf{J} \cdot \mathbf{R}_T}{eN_i} - Q_\Delta. \tag{33}$$

The collisional transfer of energy between electrons and ions is given by

$$Q_\Delta = \frac{3 m_e N_i}{m_i \tau_e} (T_e - T_i). \tag{34}$$

It is assumed that all the Joule heating goes into the electrons. The particle transport equation is used to change Eq. (31) into an equation for $\partial P_e/\partial t$. The equation for $\partial P_e/\partial t$ is integrated over a surface of constant ρ. After a few steps similar to the ones used in the last section, the surface averaged transport equation for P_e becomes

$$\begin{aligned} \frac{d}{dt}(P_e S_1) = &-\frac{2}{3} \frac{d}{d\rho} \left\{ \int \frac{d^2r}{|\nabla\rho|} \nabla\rho \cdot \left[\mathbf{q}_e + \frac{5}{2} P_e (\mathbf{u}_e - \mathbf{V}_\rho\right]\right\} \\ &+ \frac{2}{3} \frac{dP_e}{d\rho} \int \frac{d^2r}{|\nabla\rho|} \nabla\rho \cdot (\mathbf{u}_e - \mathbf{V}_\rho) - \frac{2}{3} P_e \frac{d}{d\rho} \int \frac{d^2r}{|\nabla\rho|} \mathbf{V}_\rho \cdot \nabla\rho \\ &+ \frac{2}{3} \int \frac{d^2r}{|\nabla\rho|} Q_e. \end{aligned} \tag{35}$$

The $\mathbf{V}_\rho \cdot \nabla\rho$ term on the right-hand side can be changed to a dS_1/dt term and combined with the left-hand side to give a transport equation for $P_e S_1^{5/3}$.

The second term on the right-hand side of Eq. (35) which represents the energy change due to work done by a flow against a pressure gradient, can be written in term of transport coefficients from the particle transport equation

$$\frac{2}{3}\frac{dP_e}{d\rho}\int\frac{d^2r}{|\nabla\rho|}\nabla\rho\cdot(u_e-\mathbf{V}_\rho)$$
$$=\frac{d(A_2Q_2)}{d\rho}\sum_{k=1}^{4}\frac{2}{3}\frac{B_{1,k}}{N_e}\frac{d}{|d\rho|}(A_kQ_k)+\frac{d(A_2Q_2)}{d\rho}\frac{2}{3}D_1. \tag{36}$$

The right-hand side of the above equation contains terms which are the product of two gradient terms. The first term on the right-hand side of this equation can be written in terms of a double sum over gradients and a symmetric matrix $s^W_{m,k}$

$$\frac{d}{d\rho}(A_3Q_3)\sum_{k=1}^{4}\frac{2}{3}\frac{B_{1,k}}{N_e}\frac{d}{d\rho}(A_kQ_k)=\sum_{m=1}^{4}\frac{d}{d\rho}(A_mQ_m)\left[\sum_{k=1}^{4}s^W_{m,k}\frac{d}{d\rho}(A_kQ_k)\right]. \tag{37}$$

This term is in a symmetric form. When it is differenced, the derivatives in the square brackets will be treated explicitly and the other implicitly. This is equivalent to treating each gradient in this term as half implicit and half explicit. The non-zero elements of the $s^W_{m,k}$ matrix are,

$$s^W_{2,1}=s^W_{1,2}=\frac{B_{1,1}}{3N_i}, \tag{38a}$$

$$s^W_{2,2}=\frac{2B_{1,2}}{3N_i}, \tag{38b}$$

$$s^W_{2,3}=s^W_{3,2}=\frac{B_{1,3}}{3N_i}, \tag{38c}$$

$$s^W_{2,4}=s^W_{4,2}=\frac{B_{1,4}}{3N_i}. \tag{38d}$$

The terms in Eq. (35) which contains Q_e also contains terms which are products of two gradients, since Q_e is proportional to currents squared, and the currents are proportional to the gradients. This term can be written as

$$\frac{2}{3}\int\frac{d^2r}{|\nabla\rho|}Q_e=\sum_{m=1}^{4}\sum_{k=1}^{4}\frac{d}{d\rho}(A_mQ_m)\,s^J_{m,k}\frac{d}{d\rho}(A_kQ_k)-\frac{2}{3}\int\frac{d^2r}{|\nabla\rho|}Q_\Delta. \tag{39}$$

Where the non-zero elements of the $s^J_{m,k}$ matrix are

$$s^J_{2,1}=s^J_{1,2}=\left(\frac{c^2}{\psi_\delta^2}\right)\frac{S_3T_e}{2\sigma_\perp}, \tag{40a}$$

$$s^J_{3,1}=s^J_{1,3}=\left(\frac{c^2}{\psi_\delta^2}\right)\frac{S_3T_e}{2\sigma_\perp}, \tag{40b}$$

$$s^J_{2,2} = \left(\frac{c^2}{\psi_\delta^2}\right)\frac{2}{3}\left(\frac{f^2 S_4}{\sigma_{||}} - \frac{S_3}{2\sigma_\perp}\right), \tag{40c}$$

$$s^J_{2,3} = s^J_{3,2} = \left(\frac{c^2}{\psi_\delta^2}\right)\frac{1}{3}\left(\frac{2f^2 S_4}{\sigma_{||}} + \frac{S_3}{2\sigma_\perp}\right), \tag{40d}$$

$$s^J_{2,4} = s^J_{4,2} = \left(\frac{c^2}{\psi_\delta^2}\right)\frac{1}{3}\frac{fS_1}{2\pi\sigma_{||}}, \tag{40e}$$

$$s^J_{3,3} = \left(\frac{c^2}{\psi_\delta^2}\right)\frac{2}{3}\left(\frac{f^2 S_4}{\sigma_{||}} + \frac{S_3}{\sigma_\perp}\right), \tag{40f}$$

$$s^J_{3,4} = s^J_{4,3} = \left(\frac{c^2}{\psi_\delta^2}\right)\frac{fS_1}{6\pi\sigma_{||}}, \tag{40g}$$

$$s^J_{4,4} = \left(\frac{c^2}{\psi_\delta^2}\right)\frac{2}{3(4\pi)^2\,\sigma_{||}}\,(\psi_\delta^2 S_5 + f^2 S_2). \tag{40h}$$

The surface integral S_5 is

$$S_5 = \int \frac{d^2r}{|\nabla\rho|}\frac{|\nabla\rho|^2}{r^2}. \tag{41}$$

The electron entropy equation can now be written as

$$\begin{aligned}\frac{1}{S_1^{2/3}}\frac{d}{dt}(P_e S_1^{5/3}) = &-\frac{d}{d\rho}\left\{\frac{2P_e c^2 S_3}{3\sigma_\perp \psi_\delta^2}\left(4.66 T_e \frac{dN_e}{d\rho} - 3.16\frac{dP_e}{d\rho} + \frac{3}{2}\frac{dP_i}{d\rho}\right)\right.\\ &\left.+\frac{5}{3}P_e\left(\sum_{k=1}^{4} B_{1,k}\frac{d}{d\rho}(A_k Q_k) + \frac{S_1}{\psi_\delta}\left[\dot\psi_m(1-\rho) + \dot\psi_b \rho\right]\right)\right\}\\ &+\sum_{m=1}^{4}\sum_{k=1}^{4}\frac{d}{d\rho}(A_m Q_m)(s^W_{m,k} + s^J_{m,k})\frac{d}{d\rho}(A_k Q_k)\\ &+\frac{d}{d\rho}(A_2 Q_2)\frac{2}{3}\frac{S_1}{\psi_\delta}\left[\dot\psi_m(1-\rho) + \dot\psi_b\rho\right] - \frac{2e^2 N_i S_1}{m_i \sigma_\perp}(P_e - P_i).\end{aligned} \tag{42}$$

D. *Ion Entropy Equation*

The flux surface averaged ion entropy transport equation is derived from the ion temperature equation given by Braginskii,

$$\frac{3}{2}N_i\left(\frac{\partial T_i}{\partial t} + u_i\cdot\nabla T_i\right) + P_i\nabla\cdot u_i = -\nabla\cdot \mathbf{q}_i + Q_i, \tag{43}$$

where q_i and Q_i are given by

$$\mathbf{q}_i = -\frac{2N_i T_i}{m_i \omega_i^2 \tau_i} \nabla_\perp T_i + \frac{5}{2} \frac{cN_i T_i}{eB} \mathbf{b} \times \nabla T_i, \tag{44}$$

$$Q_i = Q_\Delta. \tag{45}$$

The derivation of the ion entropy equation is similar to the derivation of the electron entropy equation, the result is

$$\begin{aligned}
\frac{1}{S_1^{2/3}} \frac{d}{dt} (P_i S_1^{5/3}) = &-\frac{d}{d\rho} \left\{ \frac{2}{3} \frac{c^2}{\psi_\delta^2} \frac{2^{1/2} P_i S_3}{\sigma_\perp} \left(\frac{m_i}{m_e}\right)^{1/2} \left(\frac{T_e}{T_i}\right)^{3/2} \left(\frac{dP_i}{d\rho} - T_i \frac{dN_i}{d\rho}\right) \right. \\
&\left. + \frac{5}{3} P_i \left(\sum_{k=1}^{4} \frac{B_{1,k}}{N_i} \frac{d}{d\rho} (A_k Q_k) + \frac{S_1}{\psi_\delta} [\dot{\psi}_m (1-\rho) + \dot{\psi}_b \rho] \right) \right\} \\
&+ \sum_{m=1}^{4} \sum_{k=1}^{4} \frac{d}{d\rho} (A_m Q_m) s_{m,k}^i \frac{d}{d\rho} (A_k Q_k) \\
&+ \frac{d}{d\rho} (A_3 Q_3) \frac{2}{3} \frac{S_1}{\psi_\delta} [\dot{\psi}_m (1-\rho) + \dot{\psi}_b \rho] + \frac{2e^2 N_i S_1}{m_i \sigma_\perp} (P_e - P_i).
\end{aligned} \tag{46}$$

The non-zero elements of the $s_{m,k}^i$ matrix are

$$s_{3,1}^i = s_{1,3}^i = \frac{B_{1,1}}{3N_i}, \tag{47a}$$

$$s_{3,2}^i = s_{2,3}^i = \frac{B_{1,2}}{3N_i}, \tag{47b}$$

$$s_{3,3}^i = \frac{2B_{1,3}}{3N_i}, \tag{47c}$$

$$s_{3,4}^i = s_{4,3}^i = \frac{B_{1,4}}{3N_i}. \tag{47d}$$

E. *Toroidal Magnetic Flux Transport Equation*

The transport equation for the toroidal magnetic flux is derived from the toroidal component of Faraday's Law,

$$\frac{1}{r^2} \frac{\partial f}{\partial t} = -c\nabla \cdot (\mathbf{E} \times \nabla\theta). \tag{48}$$

This equation is changed to d/dt, and integrated over a flux surface. The component of Ohm's Law parallel to the magnetic field is used to evaluate $\mathbf{E} \cdot \mathbf{B}$ in terms of $J_{||}$. The resulting equation for fS_2 is

$$\frac{d}{dt}(fS_2) = -\frac{d}{d\rho}\left\{-\frac{c^2}{\sigma_{\|}\psi_\delta^2}\left[fS_1\left(\frac{dP_e}{d\rho}+\frac{dP_i}{d\rho}\right)+\left(\frac{\psi_\delta^2 S_5 + f^2 S_2}{4\pi}\right)\frac{df}{d\rho}\right]\right.$$
$$\left. + \frac{fS_2}{\psi_\delta}\left[\dot{\psi}_m(1-\rho)+\dot{\psi}_b\rho\right]\right\}. \tag{49}$$

The quantity advanced in time by this equation, fS_2, is related to the magnetic stability safety factor, q, by

$$q = \frac{fS_2}{(2\pi)^2\,\psi_\delta}. \tag{50}$$

F. *Equation for* ψ_m

Transport equations have been given for the four quantities Q_1, Q_2, Q_3 and Q_4. The sum of Q_2 and Q_3 which is related to the total pressure and Q_4 which is related to f are needed to compute the new equilibrium. Another adiabatic quantity which is needed for the equilibrium calculation is the values of ψ on each surface. The value of ψ on each surface is determined by the values of ρ on each surface, ψ_m and ψ_b. The coordinate ρ is independent of time. The value of ψ at the separatrix, ψ_b, is set to zero for all times. An equation for the evolution of ψ_m is needed to determined ψ on each surface.

To obtain an equation for ψ_m Eq. (26) is evaluated on the magnetic axis, $\rho = 0$. At this point $\nabla\psi = 0$.

$$\dot{\psi}_m = -\left(cf\frac{\mathbf{E}\cdot\mathbf{B}}{B^2}\right)_{\rho=0}. \tag{51}$$

The right-hand side of this eqution is evaluated using the component of the momentum equation parallel to **b**. The resulting equation for $\dot{\psi}_m$ is

$$\dot{\psi}_m = \frac{c^2}{\psi_\delta\sigma_{\|}}\left(\frac{f}{4\pi}\frac{df}{d\rho}+r^2\frac{dP_e}{d\rho}+r^2\frac{dP_i}{d\rho}\right)_{\rho=0}. \tag{52}$$

This equation is evaluated at each time step to obtain a new value of ψ_m.

G. *Summary of Transport Coefficients*

In this section the classical transport coefficients, $B_{l,k}$, C_l, D_l, $E_{l,k}$, $F_{l,k}$ and G_l, are given for each of the four equations $l = 1, 2, 3$ and 4. These coefficients go into the transport equation of the form given by Eq. (8). The coefficients which are zero are not listed.

$l = 1$ particle transport coefficients:

$$B_{1,1} = -N_i\frac{c^2}{\psi_\delta^2}\frac{3}{2}\frac{T_e S_2}{\sigma_\perp}, \tag{53a}$$

$$B_{1,2} = -N_i \frac{c^2}{\psi_\delta^2} \left(\frac{f^2 S_4}{\sigma_{||}} - \frac{S_3}{2\sigma_\perp} \right), \tag{53b}$$

$$B_{1,3} = -N_i \frac{c^2}{\psi_\delta^2} \left(\frac{f^2 S_4}{\sigma_{||}} + \frac{S_3}{\sigma_\perp} \right), \tag{53c}$$

$$B_{1,4} = -N_i \frac{c^2}{\psi_\delta^2} \frac{f S_1}{4\pi\sigma_{||}}, \tag{53d}$$

$$D_1 = \frac{S_1}{\psi_\delta} \left[\dot{\psi}_m (1-\rho) + \dot{\psi}_b \rho \right]. \tag{53e}$$

$l = 2$ electron entropy transport coefficients:

$$B_{2,1} = \frac{5}{3} T_e B_{1,1} + P_e \frac{2}{3} \frac{c^2}{\psi_\delta^2} \frac{S_3}{\sigma_\perp} T_e a_e 4.66, \tag{54a}$$

$$B_{2,2} = \frac{5}{3} T_e B_{1,2} + P_e \frac{2}{3} \frac{c^2}{\psi_\delta^2} \frac{S_3}{\sigma_\perp} \left(\frac{3}{2} - a_e 4.66 \right), \tag{54b}$$

$$B_{2,3} = \frac{5}{3} T_e B_{1,3} + P_e \frac{c^2}{\psi_\delta^2} \frac{S_3}{\sigma_\perp}, \tag{54c}$$

$$B_{2,4} = \frac{5}{3} T_e B_{1,4}, \tag{54d}$$

$$D_2 = \frac{5}{3} D_1, \tag{54e}$$

$$E_{2,1} = \sum_{k=1}^{4} (s_{1,k}^W + s_{1,k}^J) \frac{d}{d\rho} (A_k Q_k), \tag{54f}$$

$$E_{2,2} = \sum_{k=1}^{4} (s_{2,k}^W + s_{2,k}^J) \frac{d}{d\rho} (A_k Q_k) + \frac{2}{3} D_1, \tag{54g}$$

$$E_{2,3} = \sum_{k=1}^{4} (s_{3,k}^W + s_{3,k}^J) \frac{d}{d\rho} (A_k Q_k), \tag{54h}$$

$$E_{2,4} = \sum_{k=1}^{4} (s_{4,k}^W + s_{4,k}^J) \frac{d}{d\rho} (A_k Q_k), \tag{54i}$$

$$F_{2,2} = -\frac{2e^2 N_i S_1}{m_i \sigma_\perp}, \qquad F_{2,3} = \frac{2e^2 N_i S_1}{m_i \sigma_\perp}, \tag{54j}$$

$$G_2 = G_2^{(\mathrm{rad})}. \tag{54k}$$

The coefficient a_e in equations (54a–54k) is a factor which will be set to one for classical, but will be greater than one for the simulation of anomalous electron thermal conductivity, as described in Section J. The radiation cooling term $G_2^{(\mathrm{rad})}$ is given in Section H.

$l = 3$ ion entropy transport coefficients:

$$B_{3,1} = \frac{5}{3} T_i B_{1,1} + P_i \frac{c^2}{\psi_\delta^2} \frac{2S_3}{3\sigma_\perp} \left(\frac{2m_i}{m_e}\right)^{1/2} \left(\frac{T_e}{T_i}\right)^{3/2} T_i, \tag{55a}$$

$$B_{3,2} = \frac{5}{3} T_i B_{1,2}, \tag{55b}$$

$$B_{3,3} = \frac{5}{3} T_i B_{1,3} - P_i \frac{c^2}{\psi_\delta^2} \frac{2S_3}{3\sigma_\perp} \left(\frac{2m_i}{m_e}\right)^{1/2} \left(\frac{T_e}{T_i}\right)^{3/2}, \tag{55c}$$

$$B_{3,4} = \frac{5}{3} T_i B_{1,4}, \tag{55d}$$

$$D_3 = \frac{5}{3} D_1, \tag{55e}$$

$$E_{3,1} = \sum_{k=1}^{4} s_{1,k}^i \frac{d}{d\rho} (A_k Q_k), \tag{55f}$$

$$E_{3,2} = \sum_{k=1}^{4} s_{2,k}^i \frac{d}{d\rho} (A_k Q_k), \tag{55g}$$

$$E_{3,3} = \sum_{k=1}^{4} s_{3,k}^i \frac{d}{d\rho} (A_k Q_k) + \frac{2}{3} D_1, \tag{55h}$$

$$E_{3,4} = \sum_{k=1}^{4} s_{4,k}^i \frac{d}{d\rho} (A_k Q_k), \tag{55i}$$

$$F_{3,2} = \frac{2e^2 N_i S_1}{m_i \sigma_\perp}, \qquad F_{3,3} = -\frac{2e^2 N_i S_1}{m_i \sigma_\perp}, \tag{55j}$$

$$G_3 = G_3^{(\mathrm{beam})} + G_3^{(cx)}, \tag{55k}$$

where the term due to beam heating, $G_3^{(\mathrm{beam})}$ and charge exchange loss, $G_3^{(cx)}$, have to be added; these terms will be described in Sections I and K, respectively.

$l = 4$ toroidal magnetic flux transport coefficients:

$$B_{4,2} = -\frac{c^2}{\psi_\delta^2} \frac{fS_1}{\sigma_{||}}, \tag{56a}$$

$$B_{4,3} = -\frac{c^2}{\psi_\delta^2} \frac{fS_1}{\sigma_{||}}, \tag{56b}$$

$$B_{4,4} = -\frac{c^2}{\psi_\delta^2} \frac{(\psi_\delta^2 S_5 + f^2 S_2)}{4\pi\sigma_{||}}, \tag{56c}$$

$$D_4 = \frac{S_2}{S_1} D_1. \tag{56d}$$

In the following four sections the coefficients are given for physical processes other than classical diffusion (a_e, $G_2^{(\mathrm{rad})}$, $G_3^{(\mathrm{beam})}$ and $G_3^{(cx)}$).

H. *Radiation Cooling of Electrons*

A profile of impurity ions is assumed to be given as a function of ρ. The resulting energy loss rate is given in Ref. [9], which assumes a coronal equilibrium,

$$\mathscr{P}_R = N_i N_z L_z(T_e). \tag{57}$$

Where N_z is the impurity particle density, and $L(T_e)$ is given in Ref. [9] as a polynomial in $\log(T_e)$. The radiation cooling leads to an additional G term in the electron entropy equation ($l = 2$),

$$G_2^{(\mathrm{rad})} = -\frac{2}{3} S_1 N_i N_z L_z(T_e). \tag{58}$$

I. *Neutral Beam Heating*

The calculation of the deposition of neutral beam energy uses a code developed by [illegible] density on the r, z grid are used to compute the attenuation of the beam. The energy deposition rate is then given on the r, z grid. This rate is then averaged over each flux surface to give the term which goes into the ion entropy equation.

J. *Anomalous Electron Thermal Conductivity*

Many plasma experiments indicate that the electron thermal conductivity is larger than classical. This anomalous thermal conductivity is modeled in the FRT code by multiplying the classical thermal conductivity by some given factor a_e which appears in Eqs. (54a)–(54k).

K. *Charge Exchange Loss*

A slab model is used to approximate a neutral gas profile as a function of ρ. The neutral density is used to compute a charge exchange loss term in the ion entropy equation. The neutral gas is assumed to flow into the plasma at some given thermal speed, v_n. The neutral density is computed surface by surface using an attenuation factor determined by charge exchange, ion ionization, and electron ionization cross-sections. The cross-sections for these processes are given as polynomials in Ref. [11]. The neutral density N_u on the outer flux surface is assumed to be given. The neutral density on the next flux surface inward, $N_{u,j-1}$, is computed from the density on the jth flux surface, $N_{u,j}$ by

$$N_{u,j-1} = N_{u,j} \exp\left(-\left(\frac{l_{j-1/2}\mu_{j-1/2}}{v_n}\right)\right). \tag{59}$$

The attenuation factor μ is given by

$$\mu = N_e\langle\sigma v\rangle_e + N_i\langle\sigma v\rangle_i + N_i\langle\sigma v\rangle_X, \tag{60}$$

where $\langle\sigma v\rangle_e$, $\langle\sigma v\rangle_i$ and $\langle\sigma v\rangle_X$ are respectively the electron ionization, ion ionization and charge exchange cross-sections. $l_{j+1/2}$ is an average of the distance between the surfaces of constant ρ_j and ρ_{j-1}. An approximation for this quantity can be obtained from the difference in ρ between the surfaces, $\Delta\rho_{j-1/2}$, and the S_2 and S_5.

$$l_{j-1/2} \approx \Delta\rho_{j-1/2}\left(\frac{S_{2,j-1/2}}{S_{5,j-1/2}}\right)^{1/2}. \tag{61}$$

With the neutral density given on each flux surface, the charge exchange loss term can be added to the ion entropy equation.

$$G_3^{(cx)} = -S_1 T_i N_i N_u\langle\sigma v\rangle_X. \tag{62}$$

L. *Finite Differencing of the Transport Equations*

The ρ grid is indexed by the subscript j. The o-point corresponds to $j=1$ ($\rho_1 = 0$), and the separatrix is $j = M$ ($\rho_M = 0$). A superscript n is added to denote the time level. The transport equations will be solved for $Q_{l,j}^{n+1}$ given the values $Q_{l,j}^n$. The various terms in the transport equation (8) will be differenced. The B term is

$$\begin{aligned}\frac{d}{d\rho}\left[B_{l,k}\frac{d}{d\rho}(A_k Q_k)\right] = {}& \frac{B_{l,k,j-1/2}^{n+1}}{\Delta\rho_j\Delta\rho_{j-1/2}} A_{k,j-1}^n Q_{k,j-1}^{n+1} \\ & - \left(\frac{B_{l,k,j-1/2}^{n+1}}{\Delta\rho_j\Delta\rho_{j-1/2}} + \frac{B_{l,k,j+1/2}^{n+1}}{\Delta\rho_j\Delta\rho_{j+1/2}}\right) A_{k,j}^n Q_{k,j}^{n+1} \\ & + \frac{B_{l,k,j+1/2}^{n+1}}{\Delta\rho_j\Delta\rho_{j+1/2}} A_{k,j+1}^n Q_{k,j+1}^{n+1}.\end{aligned} \tag{63}$$

The C term is differenced as

$$\frac{dC_l}{d\rho} = \frac{C_{l,j+1/2}^{n+1} - C_{l,j-1/2}^{n+1}}{\Delta\rho_j}. \tag{64}$$

A pseudo upwind method is used to difference the D term. This method is used because the convection due to ψ_m can be large near the o-point.

$$\frac{d}{d\rho}(D_l A_l Q_l) = -\frac{D^{n+1}_{l,j-1/2}\xi^{n+1}_{l,j-1/2}}{\Delta\rho_j} A^n_{l,j-1} Q^{n+1}_{l,j-1}$$
$$+\frac{1}{\Delta\rho_j}\left[D^{n+1}_{l,j+1/2}\,\xi^{n+1}_{l,j+1/2} - D^{n+1}_{l,j-1/2}(1-\xi^{n+1}_{l,j-1/2})\right] A^n_{l,j} Q^{n+1}_{l,j}$$
$$+\frac{D^{n+1}_{l,j+1/2}(1-\xi^{n+1}_{l,j+1/2})}{\Delta\rho_j} A^n_{l,j+1} Q^{n+1}_{l,j+1}. \tag{65}$$

The value of ξ is determined by the sign of D.

$$\text{if}\quad D^{n+1}_{l,j+1/2} < 0 \qquad \text{then}\quad \xi^{n+1}_{l,j+1/2} = 0, \tag{66a}$$
$$\text{if}\quad D^{n+1}_{l,j+1/2} > 0 \qquad \text{then}\quad \xi^{n+1}_{l,j+1/2} = 1. \tag{66b}$$

The E term is differenced as

$$E_{l,k}\frac{d}{d\rho}(A_k Q_k) = E^{n+1}_{l,k,j}\frac{A^n_{k,j+1} Q^{n+1}_{k,j+1} - A^n_{k,j-1} Q^{n+1}_{k,j-1}}{\Delta\rho_{j-1/2} + \Delta\rho_{j+1/2}}. \tag{67}$$

The left-hand side is differenced as

$$\frac{dQ_l}{dt} = \frac{Q^{n+1}_l - Q^n_l}{\Delta t}. \tag{68}$$

The simultaneous solution of the transport equations yields a block tridiagonal system to solve.

$$\sum_{k=1}^{4}(a_{l,k,j} Q^{n+1}_{k,j-1} + b_{l,k,j} Q^{n+1}_{k,j} + c_{l,k,j} Q^{n+1}_{k,j+1}) = d_{l,j}, \qquad l = 1, 2, 3, 4, \tag{69a}$$

where,

$$a_{l,k,j} = \mathscr{S}^n_{l,j}\left[\frac{B^{n+1}_{l,k,j-1/2}}{\Delta\rho_j\,\Delta\rho_{j-1/2}} + \frac{E^{n+1}_{l,k,j}}{(\Delta\rho_{j-1/2} + \Delta\rho_{j+1/2})} - \delta_{l,k}\left(\frac{D^{n+1}_{l,j-1/2}\xi^{n+1}_{j-1/2}}{\Delta\rho_j}\right)\right] A^n_{k,j-1}, \tag{69b}$$

$$b_{l,k,j} = \mathscr{S}^n_{l,j}\left\{-\frac{1}{\Delta\rho_j}\left(\frac{B^{n+1}_{l,k,j-1/2}}{\Delta\rho_{j-1/2}} + \frac{B^{n+1}_{l,k,j+1/2}}{\Delta\rho_{j+1/2}}\right) - F^{n+1}_{l,k,j} - \delta_{l,k}\left[-\frac{D^{n+1}_{l,j+1/2}\xi^{n+1}_{l,j+1/2}}{\Delta\rho_j} + \frac{D^{n+1}_{l,j-1/2}(1-\xi^{n+1}_{l,j-1/2})}{\Delta\rho_j}\right]\right\} A^n_{k,j} + \frac{\delta_{k,l}}{\Delta t}, \tag{69c}$$

$$c_{l,k,j} = \mathscr{S}^n_{l,j} \left[\frac{B^{n+1}_{l,k,j+1/2}}{\Delta\rho_j \Delta\rho_{j+1/2}} - \frac{E^{n+1}_{l,k,j}}{(\Delta\rho_{j-1/2} + \Delta\rho_{j+1/2})} + \delta_{l,k} \left(\frac{D^{n+1}_{l,j+1/2}(1 - \xi^{n+1}_{j+1/2})}{\Delta\rho_j} \right) \right] A^n_{k,j+1} \tag{69d}$$

and

$$d_{l,j} = \mathscr{S}^n_{l,k} \left(-\frac{C^{n+1}_{l,j+1/2} - C^{n+1}_{l,j-1/2}}{\Delta\rho_j} + G^{n+1}_{l,j} \right) + \frac{Q^n_{l,k}}{\Delta t}. \tag{69e}$$

The transport coefficients, $B_{l,k,j}$, $C_{l,j}$, $D_{l,j}$, $E_{l,k,j}$, $F_{l,k,j}$ and $G_{l,j}$, can be computed explicitly, that is, at the $n+1$ time level as shown, or implicitly, at the n time level. $\delta_{l,k}$ is the Kronecker delta function.

The block tridiagonal system has blocks of size 4 by 4. If the coefficients are computed explicitly then the solution is iterated, computing the coefficients at each iteration until some convergence criterion is satisfied. One quantity in the solution which is not computed at the $n+1$ time level is the quantities which are determined from the equilibrium calculation. These are quantities which depend on S_1, S_2, S_3, S_4 and S_5.

The boundary conditions which are used in this calculation is that the densities and temperatures are given on the separatrix. The toroidal magnetic flux function, f, is zero on the separatrix, since the surface goes to $r = 0$ and there can not be any infinite current flowing up the z axis. The boundary conditions used at he o-point is that the particle and heat fluxes are all zero.

III. Equilibrium Calculation

Two types of equilibrium calculations are used in the FRT code, a 1–D equilibrium calculation which is done after each transport calculation, and a 2–D equilibrium calculation which is done only when the adabatic quantities Q_1, Q_2, Q_3 and Q_4 have changed significantly. The 2–D equilibrium calculation will be described first.

The equilibrium is determined by the poloidal magnetic flux function $\psi(r, z)$. The central equation in the determination of ψ in an axisymmetric plasma is the Grad–Shafranov equation which is (3).

The left-hand side is an elliptic operator which would imply boundary conditions on the boundary of the r, z grid. The right-hand side looks like an ordinary differential equation, which would imply boundary conditions at one end of the ψ range. The quantities which are computed by the transport calculation which determine the equilibrium are the adiabatic quantities, $Q_1(\rho)$, $Q_2(\rho)$, $Q_3(\rho)$, $Q_4(\rho)$ and

ψ_m. The total pressure, P, and the toroidal magnetic flux function f are given in terms of Q_2, Q_3 and Q_4 as

$$P = \frac{Q_2 + Q_3}{S_1^{5/3}}, \tag{70}$$

$$f = \frac{Q_4}{S_2}. \tag{71}$$

The S_1 and S_2 in the above equations are integrals over surfaces of constant ψ and thus depend on $\psi(r, z)$.

A method of solving Eq. (3) with Q_2, Q_3, Q_4 and ψ_m given has been given in Ref. [2]. This method consists of alternating between Eq. (3) which determines $\psi(r, z)$ and the flux surface average of Eq. (3), which is

$$\frac{d}{d\mathscr{V}}\left(K\frac{d\psi}{d\mathscr{V}}\right) = -4\pi\frac{dP}{d\psi} - f\frac{df}{d\psi}\left\langle\frac{1}{r^2}\right\rangle, \tag{72}$$

where the average of any function F is

$$\langle F\rangle = \int \frac{d^2r}{|\nabla\rho|}F \bigg/ \int \frac{d^2r}{|\nabla\rho|}. \tag{73}$$

The integrals are over surfaces of constant ρ. The volume enclosed by a surface of constant ψ is $\mathscr{V}(\psi)$. The function K is

$$K = \left\langle\frac{|\nabla\mathscr{V}|^2}{r^2}\right\rangle. \tag{74}$$

The first step in the determination of the equilibrium is guessing values of the ψ function, or using the values from the last time step. With the values of ψ given the S_1 and S_2 can be determined. With the S_1 and S_2 given the P and f can be computed, and Eq. (3) is solved for ψ in a rectangular region by an Incomplete Cholesky Conjugate Gradient (ICCG) method, [8]. It was found that the ADI (alternating direction implicit) scheme did not converge well due to the nonlinear nature of the problem. The rz grid used in the 2–D equilibrium calculation can have variable grid point spacing (this feature was not used in the calculation presented in Section V). This allows grid points to be concentrated where more accuracy is needed. Cyclic reduction methods could not be used for the 2–D calculation due to the use of variable grid spacing.

The total flux ψ is the sum of vacuum flux and plasma flux. The vacuum flux is caused by external coils and is helf constant unless the coil current changes in time. The boundary value for the 2–D solution of (3) then consist of the vacuum flux value of the boundary plus the plasma flux value on the boundary. For the case of a conducting wall at the boundary the plasma flux is zero. For an open system the plasma flux on the boundary is obtained by integrating the appropriate Green's function times the plasma current, over the plasma volume.

The computed values of ψ on the rz grid will give a value of ψ at the o-point, ψ_m. However, this value of ψ_m will not necessarily be the desired value which is determined by Eq. (52) or the initial conditions. Now Eq. (72) is solved for ψ over the range of $\mathscr{V}=0$ to $\mathscr{V}=\mathscr{V}_s$ the volume enclosed by the separatrix. The boundary conditions which are used is that $\psi_b=\psi(\mathscr{V}_s)$ and $\psi_m=\psi(0)$ are given. This equation is solved by a tridagonal method [8]. The function K is held constant during the solution of (72). The average of $1/r^2$ is also held constant during this 1–D calculation. Once $\psi(\mathscr{V})$ is known then a new S_1 can be computed by

$$S_1=-\psi_\delta\left(\frac{d\psi}{d\mathscr{V}}\right)^{-1}. \tag{75}$$

With S_1 given and a new S_2 obtained from the average of $1/r^2$, a better value of the right-hand side of Eq. (3) is determined. Equation (3) is then solved, with ψ being given on the boundary. Now a more accurate value of the ψ function is available and the integrals S_1 and S_2 are determined again. The computation, which consist of alternating between the solution of the 1–D equation (72) and the 2–D equation (3), continues until some convergence criterion is satisfied.

When only a 1–D equilibrium solution is required, only Eq. (72) is solved. This determines a new value of S_1. The other surface integrals, S_2, S_3, S_4 and S_5, which are needed in the transport calculation, are determined by assuming that they change by the same fraction as S_1 changed by.

IV. Coupling of the Transport and Equilibrium Calculation

The calculation starts with the determination of an initial equilibrium. With the equilibrium determined, the sum of Q_2 and Q_3 is given as well as Q_4 and ψ_m. At this point the parameters of the plasma are not yet completely determined. The quantity Q_1 must be initialized, which determines the prticle density, N_i. Also the way in which the total entropy, Q_2+Q_3, is divided up between ion ad electron must be specified. With the equilibrium given the quantities S_1, S_2, S_3, S_4 and S_5 as defined by Eqs. (5), (6), (29), (30) and (41) are computed. The transport equations now advance the Q_1, Q_2, Q_3, Q_4 and ψ_m to the next time value.

One of the most costly parts of the code in terms of computer time is the 2–D equilibrium calculation. This is mainly due to the fact that the 2–D calculation has many more grid points than the 1–D transport calculation. In some problems the equilibrium does not change very much during the transport calculation. In order to eliminate some unnecessary calculation, some of the equilibrium calculations only solve the 1–D equilibrium equation.

The value of the right-hand side of the surface averaged equilibrium equation is used to determine if a 2–D equilibrium needs to be computed. When a 2–D equilibrium is computed the values of the right-hand side of the 1–D equilibrium, Eq. (72), is saved at each value of ρ. After each transport calculation, the right-hand side of the 1–D equilibrium equation, (72), is computed, assuming that S_1 and S_2 have not changed. This value is compared with the value which has been saved. If the difference (squared and integrated over the volume of the plasma) of these two right-hand sides is smaller than some given value (usually a few percent), then only a 1–D equilibrium is computed. If the difference is large, then a full 2–D equilibrium is computed. When only the 1–D equilibrium is computed, only S_1 is recomputed. The old values of S_2, S_3, S_4 and S_5 are multiplied by the ratio of the new S_1 to the old S_1, to obtain new values of S_2, S_3, S_4 and S_5.

The transport, equilibrium loop is completed by advancing the time and continuing or stopping the run.

V. Example of FRT Calculation

In this section an example of the calculations performed by the FRT code will be presented. This example is a simulation of the Beta II experiment at Lawrence Livermore National Laboratory. In this experiment a plasma is injected into a cylindrical flux conserver made of copper. The code simulates the decay of the plasma and magnetic fields after the plasma is in the flux conserver. There is no neutral beam heating in this simulation.

There is assumed to be a uniform guide magnetic field of 100 G in the z direction. The value of the plasma flux on the boundary is zero so the flux due to the guide field is used as the boundary condition for the 2–D equilibrium calculation. The guide magnetic field is assumed to have penetrated the flux conserver. The magnetic fields due to the plasma are assumed to be confined by the flux conserver. These assumptions are based on the differences in the time scales. The guide field is assumed to have been on for a long time, whereas the magnetic fields due to the plasma currents exist only for a short time, and will not have time to penetrate the flux concerver.

The initial flux surfaces, surface of constant ψ, are shown in Fig. 1. The initial magnetic field at $z = 0$ is approximately 1.5 kG on the flux conserver, and approximately −3.1 kG on the axis of rotational symmetry, $r = 0$. The poloidal magnetic field is zero at $r = 24$ cm and $z = 0$. This is the o-point. The totoidal magnetic field is zero on the separatrix, a condition which must be held at all times. The initial peak value of the toroidal magnetic field is 2.9 kG near the o-point. The initial total toroidal current carried by the plasma is 154 kA.

The initial electron and ion temperatures were 5 eV in the center and 0.1 eV on the separatrix. The initial central deuterium ion density is 1.24×10^{15} cm^{-3} and 1×10^{13} cm^{-3} on the separatrix.

Oxygen impurity is assumed to exist in the plasma with a uniform density of 1×10^{13} cm^{-3}, or approximately 0.8% of the central ion density. An anomalous

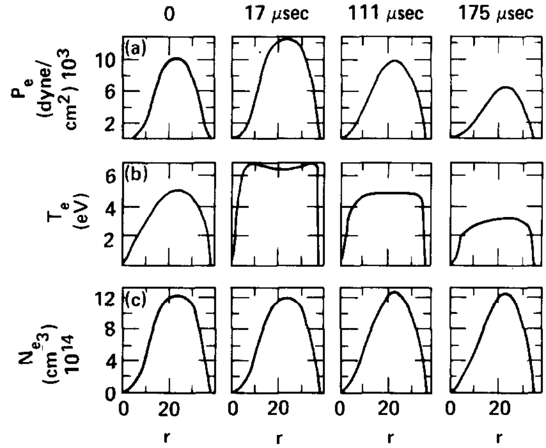

FIG. 2. Profiles (a) electron pressure, (b) electron temperature, (c) particle density. Profiles are ploted as a function of r *at* $z = 0$.

electron thermal conductivity is used which is 100 times the classical thermal conductivity. The neutral density at the separatrix, which is needed for the charge exchange ion energy loss, is $1 \times 10^{11}\ \text{cm}^{-3}$. In this calculation the charge exchange energy loss is insignificant.

Figure 2a is the profile of the electron pressure at various times. Figure 2c is the profile of the electron density at various times. Although the ion density increases a small amount at the center, the total number of particles in the plasma decreases slowly. The initial number of particles is 9.48×10^{19}, and at the end of the run, 175 μsec, the number of particles is 8.13×10^{19}.

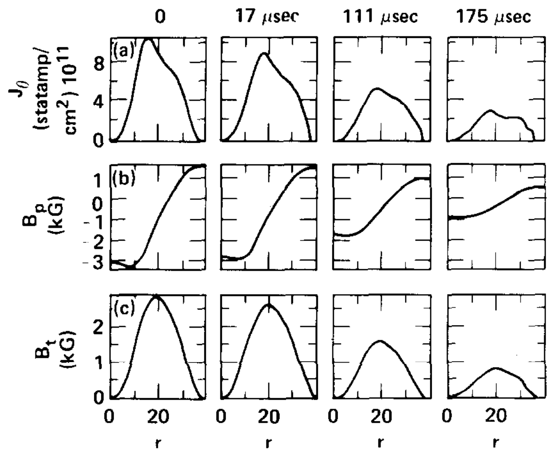

FIG. 3. Profiles (a) toroidal current density, (b) poloidal magnetic flux and (c) toroidal magnetic flux. Profiles are ploted as a function of r at $z = 0$.

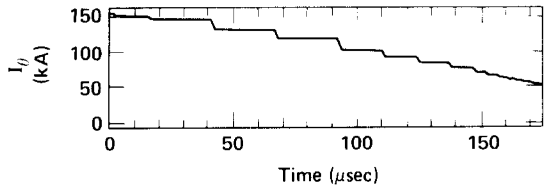

FIG. 4. Total toroidal current vs time.

The electron temperature profiles are given in Fig. 2b. The collisional tranfer of energy between electrons and ions keeps the ion temperature close to the electron temperature. They differ only by a fraction of an eV. The flattening of the temperature profile is caused by two effects. First, the thermal conductivity tends to flatten the profiles. Secondly, the peak in the current density which causes Joule heating is initially peaked at approximately $r = 16$ cm (the o-point is at 24 cm). The Joule heating term which appears in the electron entropy equation is the flux surface average of the 2–D Joule heating. The resulting 1–D heating term is peaked off the o-point. Thus the electrons are heated more off the o-point, resulting in a slight inversion of the temperature profile. The toroidal current density profiles are shown in Fig. 3a.

The profiles of the poloidal magnetic fields are given in Fig. 3b. At the end of the run the poloidal field at $r = 0$, $z = 0$ is -900 G and at the flux conserver it is slightly over 600 G.

The profiles of the toroidal magnetic field are given in Fig. 3c. The peak in the toroidal magnetic field occurs near the o-point. The peak value of the toroidal magnetic field at the end of the run is about 800 G. The magnetic stability factor, q, Eq. (50), at the o-point is initially 0.72 and at the end of the run it is 0.56. The total toroidal current carried by the plasma, which is shown in Fig. 4, has decayed from its initial value of 154 kA to 52 kA at the end of the calculation. The slight jumps in the curve are the points where the 2–D equilibrium are computed.

Figure 5 is a plot of the value of the electron temperature at the o-point vs time. There is an initial transient where the central temperatures increases from 5 to about 6.5 eV in about 15 μsec. After the transient has decayed the electron temperature is determined by a balance between Joule heating and radiation cooling due to impurities. The main flow of energy in the system is from the large reservoir of

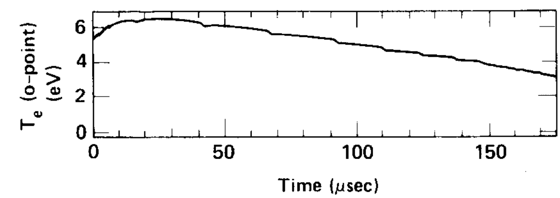

FIG. 5. Electron temperature at o-point vs time.

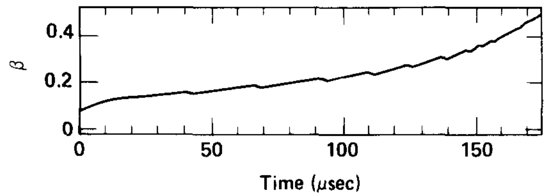

FIG. 6. β (plasma kinetic energy enclosed by separatrix divided by magnetic energy enclosed by separatrix) vs time.

magnetic field energy into the electrons via Joule heating. The electrons lose the energy by radiation. The other processes in the electron entropy equations are small, in this parameter range, compared to Joule heating and radiation cooling. In the ion temperature equation the thermal conductivity loss across the separatrix is balanced by collisional transfer of energy from the electrons. The ion and electron temperature slowly decay as the magnetic field energy reservoir is depleted. The electron and ion temperatures are about 3 eV at 175 μsec. The initial magnetic field energy is 2.66 kJ and at 175 μsec it is 0.89 kJ. The β, shown in Fig. 6 (the total plasma energy enclosed by separatrix divided by the total magnetic field energy enclosed by separatrix) is initially 0.07 and at 175 μsec it is up to 0.50.

Figure 7 is a plot of the toroidal magnetic flux (enclosed by the separatrix) as a function of time. Figure 8 is a plot of the poloidal flux (between the o-point and separatrix) as a function of time. Both the toroidal and poloidal magnetic fluxes are very close to being a linear decay in time.

There is very little change in the shape of the flux surfaces during the calculation. There is no apparent change in the position of the o-point. The crossing of the separatrix and the $z = 0$ plane is initially at 38.5 cm, and at 175 μsec it has moved in to 36.3 cm.

The dominance of the energy loss by impurity radiation is typical of the performance of the Beta II experiment [12]. The decay time constant of the magnetic field in the Beta II experiment was approximately 120 μsec [12] which is about the same as this simulation, see Figs. 7 and 8.

The number of points in the ρ grid used in this transport calculation was 21. The total number of time steps tken was 208. The 2–D equilibrium was computed 20 times during this run. The size of the r, z grid used in the 2–D equilibrium calculation

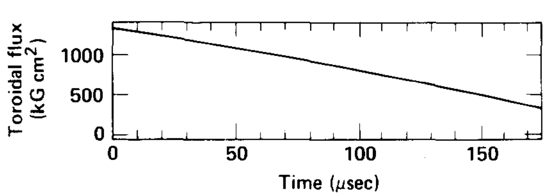

FIG. 7. Toroidal magnetic flux vs time.

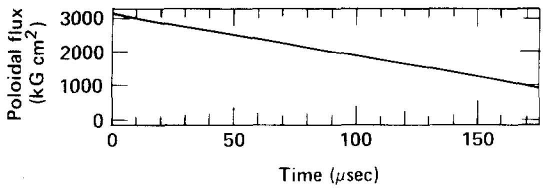

FIG. 8. Poloidal magnetic flux vs time.

was 64 by 65. Practically all of the 61 min of CRAY computer time required for this calculation was used during the 2–D equilibrium calculations. A large fraction of this time was spent doing the flux surface averages.

REFERENCES

1. I. B. BERNSTEIN, *Phys. Fluids* **17**, No. 3 (1974), 547.
2. H. GRAD, "Survey of 1 ½ D Transport Codes," Courant Institute, New York, MF–93, 1978.
3. R. N. BYRNE AND H. H. KLEIN, *J. Comput. Phys.* **26** (1978), 352.
4. J. T. HOGAN, *Nucl. Fusion* **19** (1979), 753.
5. S. P. H[illegible] AND S. C. J[illegible], *Phys. Fluids* **22** (1979), 731.
6. R. L. MILLER, *Nucl. Fusion* **20** (1980), 133.
7. S. I. BRAGINSKII, "Review of Plasma Physics," Vol. 1. Consultants Bureau, New York, 1965.
8. J. K. BOYD, S. P. AUERBACH, P. A. WILLMANN, H. L. BERK, AND B. MCNAMARA, 1980 Sherwood Meeting, Tucson, Arizona, UCRL-84044, 1980.
9. D. E. POST, R. V. JENSEN, C. B. TARTER, W. H. GRASBERGER, AND W. A. LOKKE, *At. Data Nucl. Data Tables* **20** (1977), 297.
10. J. K. BOYD, S. P. AUERBACH, H. L. BERK, B. MCNAMARA, AND P. A. WILLMANN, Sherwood Meeting, Tuscon, Arizona, 1980.
11. R. L. FREEMAND AND E. M. JONES, "Atomic Collision Processes in Plasma Physics Experiments," CLM-R137, 1974.
12. E. H. A. GRANNEMAN, G. C. GOLDENBAUM, J. H. HAMMER, C. W. HARTMAN, D. S. PRONO, J. TASKA, AND W. C. TURNER, "A Study of the Equilibrium and Decay of Compact Toroids Generated by a Magnetized Co-axial Plasma Gun," UCRL-85912, 1981.